SOME EXACT SEQUENCES FOR TOEPLITZ ALGEBRAS OF SPHERICAL ISOMETRIES

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Abstract. A family $\{T_j\}_{j\in J}$ of commuting Hilbert space operators is said to be a spherical isometry if $\sum_{j\in J} T_j^*T_j = 1$ in the weak operator topology. We show that every commuting family $\mathcal{F}$ of spherical isometries has a commuting normal extension $\hat{\mathcal{F}}$. Moreover, if $\hat{\mathcal{F}}$ is minimal, then there exists a natural short exact sequence $0 \to \mathcal{C} \to C^*(\mathcal{F}) \to C^*(\hat{\mathcal{F}}) \to 0$ with a completely isometric cross-section, where $\mathcal{C}$ is the commutator ideal in $C^*(\mathcal{F})$. We also show that the space of Toeplitz operators associated to $\mathcal{F}$ is completely isometric to the commutant of the minimal normal extension $\hat{\mathcal{F}}$. Applications of these results are given for Toeplitz operators on strictly pseudoconvex or bounded symmetric domains.

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1. Introduction

A spherical isometry on a Hilbert space $\mathcal{H}$ is a commuting family $\mathcal{S} = \{T_j\}_{j\in J}$ of bounded operators on $\mathcal{H}$ such that $\sum_{j\in J} T_j^*T_j = \mathbf{1}$, To each spherical isometry one can associate its set of Toeplitz-type operators consisting of all solutions of the operator equation

$$\sum_{j\in J} T_j^* X T_j = X.$$

One defines in a similar way Toeplitz operators associated to arbitrary commuting families of spherical isometries.

In the present paper we apply some operator space techniques in order to construct exact sequences for C^*-algebras generated by spherical isometries or by their associated Toeplitz-type operators. For this purpose we make use of the more or less known fact that the set $\mathcal{T}(\mathcal{F})$ of all Toeplitz operators associated to an arbitrary family $\mathcal{F}$ of commuting spherical isometries is an injective operator system, which means that it is the range of a completely positive unital mapping $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ with $\Phi \circ \Phi = \Phi$. This fact, when combined with a certain basic property of such projections proved in [CE77], enables us to show (see Theorem 2.9) that $\mathcal{F}$ admits a commuting normal extension $\widehat{\mathcal{F}}$ on some Hilbert space $\widehat{\mathcal{H}}$ containing $\mathcal{H}$ and that if $\widehat{\mathcal{F}}$ is minimal then there exists a *-representation π from the C^*-algebra generated by $\mathcal{T}(\mathcal{F})$ onto the commutant of $\widehat{\mathcal{F}}$ in $B(\widehat{\mathcal{H}})$ for which the compression map $\rho(X) = P_{\mathcal{H}}X|_{\mathcal{H}}$ is a complete isometric cross section whose range coincides with $\mathcal{T}(\mathcal{F})$. When we restrict π to the C^*-algebra $C^*(\mathcal{F})$ generated by $\mathcal{F}$ in $\mathcal{B}(\mathcal{H})$ we

obtain a *-representation of $C^*(\mathcal{F})$ onto $C^*(\widehat{\mathcal{F}})$ whose kernel coincides with the commutator ideal of $C^*(\mathcal{F})$. We also show that any operator in the commutant of $\mathcal{F}$ has a unique norm-preserving extension to an operator in the commutant of $\widehat{\mathcal{F}}$.

It turns out that several classes of Toeplitz operators on various Hardy spaces can be realized as common fixed points of some commuting families of completely positive mappings induced by spherical isometries. In this sense, apart from the well-known Brown-Halmos characterization of Toeplitz operators on the unit circle (see [BH63]) there is a similar result due to A.M. Davie and N. Jewell [DJ77] for the case of Toeplitz operators on the unit sphere in $\mathbb{C}^n$ where the unilateral shift is replaced by the Szego $n-$tuple. Similar characterizations also hold for Toeplitz operators on Hardy spaces on ordered groups [Mu87]. The method we shall develop here allows us to enlarge considerably the class of Toeplitz operators that admit such characterization. We show that if $\Omega \subset \mathbb{C}^n$ is either a bounded strictly pseudoconvex domain or a bounded symmetric domain and m is any Borel probability measure on the Shilov boundary of Ω then the Toeplitz operators on the corresponding Hardy space $H^2(m)$ are indeed the fixed points of a certain spherical isometry. As a consequence we obtain exact sequences and spectral inclusion theorems for operators of the type mentioned above.

2. Spherical isometries and their associated Toeplitz operators

We recall for later use the following by-product of the proof of Theorem 3.1 in [CE77], which is also stated as Lemma 6.1.2 in [ER00].

Theorem 2.1. *Let $\Phi\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a completely positive and completely contractive mapping such that $\Phi^2 = \Phi$. Then for all $X, Y \in \mathcal{B}(\mathcal{H})$ we have*

$$\Phi(\Phi(X)Y) = \Phi(X\Phi(Y)) = \Phi(\Phi(X)\Phi(Y)).$$

In [CE77] and [ER00] this result is used to show that the range of Φ is completely isometric to a C^*-algebra , where the multiplication is defined by the rule

$$\Phi(X) \circ \Phi(Y) = \Phi(\Phi(X)\Phi(Y))$$

for every $X, Y \in \mathcal{B}(\mathcal{H})$. We shall recover the latter result as a consequence of Theorem 2.2 below. To begin with, let us fix some notation. Let $\Phi\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a completely positive and completely contractive mapping such that $\Phi^2 = \Phi$ and let $\mathcal{E} = \operatorname{Ran}\Phi$ denote its range. We denote by $C^*(\mathcal{E})$ the unital C^*-algebra generated by $\mathcal{E}$ in $\mathcal{B}(\mathcal{H})$. Let

$$\Phi_0\colon C^*(\mathcal{E}) \to \mathcal{B}(\mathcal{H})$$

denote the restriction of Φ to $C^*(\mathcal{E})$. Then, according to the Stinespring dilation theorem (see for instance Theorem 5.2.1 in [ER00]), there exist a Hilbert space $\mathcal{K}$, a bounded operator $V\colon \mathcal{H} \to \mathcal{K}$ and a unital *-representation

$$\pi\colon C^*(\mathcal{E}) \to \mathcal{B}(\mathcal{K})$$

such that $\Phi_0(X) = V^*\pi(X)V$ for all $X \in C^*(\mathcal{E})$. Thus the diagram

$$\begin{array}{ccc} C^*(\mathcal{E}) & \xrightarrow{\pi} & \mathcal{B}(\mathcal{K}) \\ {\scriptstyle\Phi_0}\big\downarrow & & \big\downarrow{\scriptstyle\rho} \\ \mathcal{E} & \xrightarrow{\iota} & \mathcal{B}(\mathcal{H}) \end{array}$$

is commutative, where $\rho(X) = V^*XV$ whenever $X \in \mathcal{B}(\mathcal{K})$, and ι is the inclusion map. We shall assume that π is minimal in the sense that $\mathcal{K}$ is the smallest invariant subspace for π containing the range of V. Now, under these conditions, we can state the following theorem which will be very useful for our study of Toeplitz algebras.

Theorem 2.2. *Let $\Phi, \Phi_0, \mathcal{E}, C^*(\mathcal{E}), \mathcal{K}, V$ and π be as above. Then* $\operatorname{Ker}\Phi_0 = \operatorname{Ker}\pi$ *and the mapping*

$$\rho: \pi(C^*(\mathcal{E})) \to \mathcal{B}(\mathcal{H})$$

defined by $\rho(\pi(X)) = V^*\pi(X)V$ for $X \in C^*(\mathcal{E})$ is a complete isometry whose range equals $\mathcal{E} = \operatorname{Ran}\Phi$. Moreover, if $\operatorname{Ran}\Phi$ is σ-weakly closed, then $\pi(C^*(\mathcal{E}))$ is also σ-weakly closed, hence a von Neumann subalgebra of $\mathcal{B}(\mathcal{K})$ and the map ρ defined above is a σ-weak homeomorphism.

Proof. First of all, one can easily see that $\operatorname{Ker}\Phi_0$ is an ideal in $C^*(\mathcal{E})$. Indeed, Theorem 2.1 implies that $\operatorname{Ker}\Phi_0$ is invariant under multiplication by elements in $\mathcal{E}$ and then use the fact that, since $\mathcal{E}$ is selfadjoint, the C^*-algebra $C^*(\mathcal{E})$ is the closed linear span of all finite products of elements from $\mathcal{E}$ and the identity. Now, in order to prove the equality of the two kernels, we fix $T \in C^*(\mathcal{E})$ such that $\Phi_0(T) = 0$ and let $X, Y \in C^*(\mathcal{E})$ and $\xi, \eta \in \mathcal{H}$ be arbitrary. Then

$$(\pi(T)\pi(X)V\xi, \pi(Y)V\eta) = (V^*\pi(Y^*TX)V\xi, \eta) = (\Phi_0(Y^*TX)\xi, \eta) = 0$$

because $\operatorname{Ker}\Phi_0$ is an ideal. Since π is minimal, this shows that $\pi(T) = 0$. Since the other inclusion is trivial, the equality of the two kernels is proved.

We now show that the mapping ρ is a complete isometry. First, we see that, since $\Phi_0 = \rho \circ \pi$ and $Ker\pi = Ker\Phi_0$ it follows that ρ is one-to-one. Moreover, since $\Phi_0^2 = \Phi_0$, we have that $\rho \circ \pi \circ \rho \circ \pi = \rho \circ \pi$ hence $\pi \circ \rho$ is the identity on $\pi(C^*(\mathcal{E}))$. It then follows, since both π and ρ are completely contractive, that ρ is actually completely isometric. The last assertion of the theorem follows easily from the previous one and the separate weak*-continuity of the multiplication on a von Neumann algebra. The proof of the theorem is completed. $\square$

As we have mentioned above, this result offers an alternate proof of Theorem 3.1 in [CE77].

Theorem 2.3. *Let $\Phi: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a completely positive and completely contractive mapping such that $\Phi^2 = \Phi$. Then* $\operatorname{Ran}\Phi$ *is completely isometric with a unital C^*-algebra where the product is defined by the rule*

$$\Phi(X) \circ \Phi(Y) = \Phi(\Phi(X)\Phi(Y))$$

for all $X, Y \in \mathcal{B}(\mathcal{H})$.

Proof. Using the notation in Theorem 2.2, we see that the above defined product is precisely the one induced from $\pi(C^*(\mathcal{E}))$ via the complete isometry ρ. $\square$

Remark 2.4. It is a well-known fact that if A is a unital C^*-algebra and $\theta : A \to B(\mathcal{H})$ is a unital completely isometric mapping, then there exists a *-homomorphism

$$\pi : C^*(\theta(A)) \to A$$

such that $\pi \circ \theta = id_A$, (see Theorem 4.1 in [CE76]). Therefore, in the case when the mapping Φ in Theorem 2.2 is unital and assuming Theorem 2.3 one can immediately

see that the mapping Φ_0 appearing in Theorem 2.2 becomes a *-homomorphism when its range is endowed with the multiplication defined in Theorem 2.3. This offers a shorter proof of Theorem 2.2; however the line we took in that theorem gives simultaneously the isomorphism in Theorem 2.3 and a spatial representation for that algebraic structure, that will be useful in the sequel. □

We shall apply Theorem 2.2 to the study of the C^*- algebra generated by a commuting family of spherical isometries.

Definition 2.5. A commuting family $\mathcal{S} = \{T_j\}_{j\in J}$ of bounded operators on a Hilbert space $\mathcal{H}$ is said to be a *spherical isometry* if

$$\sum_{j\in J} T_j^* T_j = 1$$

in the weak operator topology. □

For instance, if m is any probability Borel measure on the unit sphere S^{2n-1} in $\mathbb{C}^n$, and $\mathcal{H} \subset L^2(m)$ is a jointly invariant subspace for the multiplication operators $\{M_{z_1}, \ldots, M_{z_n}\}$ on $L^2(m)$, then their restrictions $\{T_{z_1}, \ldots, T_{z_n}\}$ to $\mathcal{H}$ form a spherical isometry. Of particular interest is the case when m is the normalized area measure and $\mathcal{H}$ is the $L^2(m)$-closure of all analytic polynomials (the Hardy space $H^2(S^{2n-1})$) in which case $\{T_{z_1}, \ldots, T_{z_n}\}$ is called the Szego n-tuple on $H^2(S^{2n-1})$.

If $\{T_j\}_{j\in J}$ is an arbitrary family of operators on $\mathcal{H}$ satisfying the above equation in particular a spherical isometry then a completely positive unital, hence completely contractive mapping $\phi: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ can be defined by the formula

$$\phi(X) = \sum_{j\in J} T_j^* X T_j$$

and is also obvious that ϕ is σ-weakly continuous.

Our main object of study in what follows is a commuting family of spherical isometries $\mathcal{F} = \{\mathcal{S}_\alpha\}_{\alpha\in\Gamma}$ on some Hilbert space $\mathcal{H}$. This means that if $\alpha \in \Gamma$ then $\mathcal{S}_\alpha = \{T_{j,\alpha}\}_{j\in J_\alpha}$ is a spherical isometry and the union $\cup_{\alpha\in\Gamma}\mathcal{S}_\alpha$ is a commutative set of operators.

Definition 2.6. Given a commuting family $\mathcal{F} = \{\mathcal{S}_\alpha\}_{\alpha\in\Gamma}$ of spherical isometries on $\mathcal{H}$ we define, using the notations above, the space $\mathcal{T}(\mathcal{F})$ of all *$\mathcal{F}$-Toeplitz operators* to be the set of all operators $X \in \mathcal{B}(\mathcal{H})$ such that

$$\sum_{j\in J_\alpha} T_{j,\alpha}^* X T_{j,\alpha} = X$$

for all $\alpha \in \Gamma$. □

In other words, $\mathcal{T}(\mathcal{F})$ is the set of all common fixed points of the completely positive mappings associated to each spherical isometry from $\mathcal{F}$. It is obvious that $\mathcal{T}(\mathcal{F})$ contains the commutant of $\mathcal{F}$ in particular it contains all the sets $\mathcal{S}_\alpha$ for $\alpha \in \Gamma$. We shall construct a completely positive projection Φ on this space which will play a crucial role for our study of Toeplitz algebras associated to spherical isometries. In order to do this, we need the following lemma which is a particular case of a more general result proved in [BP05]. However for completeness we shall give below a direct proof.

Lemma 2.7. *Let $\{\phi_\alpha\}_{\alpha\in\Gamma}$ be a set of commuting completely positive unital and σ-weak continuous mappings acting on $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Then there exists a completely positive mapping $\Phi\colon\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{H})$ whose range is precisely the set*

$$\{X\in\mathcal{B}(\mathcal{H})\colon\phi_\alpha(X)=X,\ \alpha\in\Gamma\}$$

and such that $\Phi^2=\Phi$.

Proof. Let S denote the semigroup of all finite products of elements from the set $\{\phi_\alpha\}_{\alpha\in\Gamma}$. Each element $s\in S$ corresponds to a completely positive unital and $\sigma-$weak continuous mapping $\psi_s\colon\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{H})$ which is a finite product of ϕ_α's. It is obvious that the fixed point set of $\{\phi_\alpha\}_{\alpha\in\Gamma}$ is the same as that of $\{\psi_s\}_{s\in S}$. We thus obtain an action

$$\gamma\colon S\times\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{H})$$

defined by

$$\gamma(s,X)=\psi_s(X)$$

for all $s\in S$ and $X\in\mathcal{B}(\mathcal{H})$. Since S is commutative, a well-known result of Dixmier [Di50] shows that S is amenable, which means that there exists a state μ on the C^*-algebra $\ell^\infty(S)$ of all bounded complex functions on S which is invariant under all translations with elements from S. More precisely, if $t\in S$ and $L_t\colon\ell^\infty(S)\to\ell^\infty(S)$ is defined by $L_t(f)(s)=f(ts)$ for $s\in S$ then $\mu(L_tf)=\mu(f)$ for all $f\in\ell^\infty(S)$.

Now, given $T\in\mathcal{B}(\mathcal{H})$, for each pair of vectors $\xi,\eta\in\mathcal{H}$ define $[\xi,\eta]_T=\mu(\gamma(\cdot,T)\xi,\eta)$ and observe that this is a bounded sesquilinear map therefore there exists an operator that we shall denote by $\Phi(T)$ in $\mathcal{B}(\mathcal{H})$ such that

$$(\Phi(T)\xi,\eta)=[\xi,\eta]_T$$

for all $\xi,\eta\in\mathcal{H}$. It is now a matter of routine to verify that the mapping $T\mapsto\Phi(T)$ is completely positive. It is straightforward to see that if $T\in\mathcal{B}(\mathcal{H})$ is such that $\psi_s(T)=T$ for all $s\in S$ then $\Phi(T)=T$ as well.

We will show now that $\psi_s(\Phi(T))=\Phi(T)$ for all $T\in\mathcal{B}(\mathcal{H})$. In order to see that, recall that all the mappings ψ_s are σ-weakly continuous, which means that for any $s\in S$ there exists a norm continuous mapping ψ_s^* on the space $\mathfrak{S}_1(\mathcal{H})$ of trace-class operators on $\mathcal{H}$ such that $\mathrm{tr}(\psi_s(T)L)=\mathrm{tr}(T\psi_s^*(L))$ for all $T\in\mathcal{B}(\mathcal{H})$ and $L\in\mathfrak{S}_1(\mathcal{H})$. Moreover one can see that the mapping Φ satisfies the identity $\mathrm{tr}(\Phi(T)L)=\mu(\mathrm{tr}(\gamma(\cdot,T)L)$ for all $T\in\mathcal{B}(\mathcal{H})$ and $L\in\mathfrak{S}_1(\mathcal{H})$. It then follows that for any $t\in S$ we have

$$\mathrm{tr}(\psi_t(\Phi(T))L)=\mathrm{tr}(\Phi(T)\psi_t^*(L))=\mu(\mathrm{tr}(\gamma(\cdot,T)\psi_t^*(L))=\mu(\mathrm{tr}(\gamma(t\cdot,T)L))$$

and because μ is an invariant mean, this last term equals $\mu(\mathrm{tr}(\gamma(\cdot,T)L))$ which is equal to $\mathrm{tr}(\Phi(T)L)$ for all $T\in\mathcal{B}(\mathcal{H})$ and $L\in\mathfrak{S}_1(\mathcal{H})$. This shows that indeed $\psi_t(\Phi(T)=\Phi(T)$ for all $T\in\mathcal{B}(\mathcal{H})$, hence $\Phi^2=\Phi$ as well. The proof of this lemma is completed.

In the proof of our main result we also need the following easy lemma.

Lemma 2.8. *Suppose $\mathcal{M}\subset\mathcal{B}(\mathcal{H})$ is a von Neumann algebra and $\mathcal{M}'$ denotes its commutant. Let $\mathcal{R}=\{T_j\}_{j\in J}$ be a family of operators in $\mathcal{M}$ such that for all $X\in\mathcal{M}$*

$$\sum_{j\in J}T_j^*XT_j=X$$

in the $\sigma-$weak topology. Then $\mathcal{R} \subset \mathcal{M} \cap \mathcal{M}'$.

The main result of this paper is the following.

Theorem 2.9. *Let $\mathcal{F} = \{\mathcal{S}_\alpha\}_{\alpha\in\Gamma}$ be a commuting family of spherical isometries on some Hilbert space $\mathcal{H}$ with $\mathcal{S}_\alpha = \{T_{j,\alpha}\}_{j\in J_\alpha}$ for each $\alpha \in \Gamma$ and let $\mathcal{T}(\mathcal{F})$ be the space of all $\mathcal{F}$-Toeplitz operators (see* Definition 2.6 *above). Let also $C^*(\mathcal{T}(\mathcal{F}))$ denote the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\mathcal{T}(\mathcal{F})$. Then we have:*

(1) *There exists a completely positive unital mapping $\Phi: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that $\Phi^2 = \Phi$ and whose range coincides with $\mathcal{T}(\mathcal{F})$.*
(2) *There exist a Hilbert space $\widehat{\mathcal{H}}$ containing $\mathcal{H}$ and a commuting family $\widehat{\mathcal{F}} = \{\widehat{\mathcal{S}}_\alpha\}_{\alpha\in\Gamma}$ of normal spherical isometries on $\widehat{\mathcal{H}}$ with $\widehat{\mathcal{S}}_\alpha = \{\widehat{T}_{j,\alpha}\}_{j\in J_\alpha}$ which leaves $\mathcal{H}$ invariant and whose restriction to $\mathcal{H}$ coincides with $\mathcal{F}$ in other words the family $\mathcal{F}$ is subnormal.*
(3) *Suppose that the normal extension $\widehat{\mathcal{F}}$ is minimal, i.e. $\widehat{\mathcal{H}}$ is the smallest reducing subspace for $\widehat{\mathcal{F}}$ containing $\mathcal{H}$. Then there exists a unital *-representation*
$$\pi: C^*(\mathcal{T}(\mathcal{F})) \to \mathcal{B}(\widehat{\mathcal{H}})$$
such that:
(3a) *$\pi(T_{j,\alpha}) = \widehat{T}_{j,\alpha}$ for all $\alpha \in \Gamma$ and $j \in J_\alpha$.*
(3b) *If $P_\mathcal{H}$ is the orthogonal projection of $\widehat{\mathcal{H}}$ onto $\mathcal{H}$ then*
$$\Phi(X) = P_\mathcal{H}\pi(X)|_\mathcal{H}$$
for every $X \in C^*(\mathcal{T}(\mathcal{F}))$.
(3c) *The image $\pi(C^*(\mathcal{T}(\mathcal{F})))$ of π coincides with the commutant in $\mathcal{B}(\widehat{\mathcal{H}})$ of the C^*-algebra $C^*(\widehat{\mathcal{F}})$ generated by $\widehat{\mathcal{F}}$ in $\mathcal{B}(\widehat{\mathcal{H}})$.*
(3d) *The mapping*
$$\rho: \pi(C^*(\mathcal{T}(\mathcal{F}))) \to \mathcal{B}(\mathcal{H})$$
defined by $\rho(\pi(X)) = P_\mathcal{H}\pi(X)|_\mathcal{H}$ for $X \in C^*(\mathcal{T}(\mathcal{F}))$ is a complete isometry onto the space $\mathcal{T}(\mathcal{F})$ of all $\mathcal{F}$-Toeplitz operators such that $\pi\circ\rho$ is the identity on $\pi(C^*(\mathcal{T}(\mathcal{F})))$. Therefore the short exact sequence
$$0 \to \operatorname{Ker}\pi \hookrightarrow C^*(\mathcal{T}(\mathcal{F})) \xrightarrow{\pi} \pi(C^*(\mathcal{T}(\mathcal{F}))) \to 0$$
has a completely isometric cross section. Moreover, $Ker\pi$ coincides with the closed two-sided ideal of $C^*(\mathcal{T}(\mathcal{F}))$ generated by all operators of the form $XY - \Phi(XY)$ with $X, Y \in \mathcal{T}(\mathcal{F})$.
(3e) *If $C^*(\mathcal{F})$ denotes the unital C^*-algebra generated by $\mathcal{F}$ in $\mathcal{B}(\mathcal{H})$ then $\Phi(C^*(\mathcal{F})) = C^*(\mathcal{F}) \cap \mathcal{T}(\mathcal{F})$. Moreover the kernel of the restriction of π to $C^*(\mathcal{F})$ coincides with the closed ideal of $C^*(\mathcal{F})$ generated by all the commutators $XY - YX$ with $X, Y \in \mathcal{T}(\mathcal{F}) \cap C^*(\mathcal{F})$ hence it coincides with the commutator ideal $\mathcal{C}$ of $C^*(\mathcal{F})$. Therefore we have a short exact sequence*
$$0 \to \mathcal{C} \hookrightarrow C^*(\mathcal{F}) \xrightarrow{\pi} C^*(\widehat{\mathcal{F}}) \to 0$$
for which the restriction of ρ to $C^*(\widehat{\mathcal{F}})$ is a completely isometric cross section.

(3f) *An operator $X \in \mathcal{B}(\mathcal{H})$ belongs to the commutant of $\mathcal{F}$ if and only if both X and X^*X belong to the space $\mathcal{T}(\mathcal{F})$ of $\mathcal{F}$-Toeplitz operators. In this case there exists a unique operator $\hat{X}$ in the commutant of $\widehat{\mathcal{F}}$ which leaves $\mathcal{H}$ invariant and whose restriction to $\mathcal{H}$ coincides with X. Moreover the map $X \mapsto \hat{X}$ is norm preserving.*

Proof. For each $\alpha \in \Gamma$ let $\phi_\alpha \colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be the completely positive σ-weakly continuous mapping associated to the spherical isometry $\mathcal{S}_\alpha$, so for all $X \in \mathcal{B}(\mathcal{H})$ we have $\phi_\alpha(X) = \sum_{j \in J_\alpha} T^*_{j,\alpha} X T_{j,\alpha}$. It follows that $\mathcal{T}(\mathcal{F})$ is precisely the set of common fixed points of the commuting family of mappings $\{\phi_\alpha\}_{\alpha\in\Gamma}$. Therefore we can apply Lemma 2.7 to infer the existence of an idempotent completely positive mapping $\Phi \colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ whose range is precisely $\mathcal{T}(\mathcal{F})$. This proves item (1).

Let Φ as in item (1) and let Φ_0 denote its restriction to $C^*(\mathcal{T}(\mathcal{F}))$. Denote by $\pi \colon C^*(\mathcal{T}(\mathcal{F})) \to \mathcal{B}(\widehat{\mathcal{H}})$ the minimal Stinespring dilation of Φ_0. Therefore there exists an isometry $V \colon \mathcal{H} \to \widehat{\mathcal{H}}$ such that

$$\Phi_0(X) = V^*\pi(X)V$$

for all $X \in C^*(\mathcal{T}(\mathcal{F}))$. We see that we are precisely in the situation of Theorem 2.2 above, and moreover the range of Φ is also σ-weakly closed because it is the set of all common fixed points of a family of σ-weakly continuous mappings. The conclusion that follows from Theorem 2.2 is that the mapping

$$\rho \colon \pi(C^*(\mathcal{T}(\mathcal{F}))) \to \mathcal{B}(\mathcal{H})$$

defined by $\rho(\pi(X)) = V^*\pi(X)V$ for $X \in C^*(\mathcal{T}(\mathcal{F}))$ is a complete isometry onto the space of all $\mathcal{F}$-Toeplitz operators and that the image of π is a von Neumann subalgebra of $\mathcal{B}(\widehat{\mathcal{H}})$. Let

$$\widehat{T}_{j,\alpha} = \pi(T_{j,\alpha})$$

for all $\alpha \in \Gamma$ and $j \in J_\alpha$ and let also denote $\widehat{\mathcal{S}}_\alpha = \{\widehat{T}_{j,\alpha}\}_{j\in J_\alpha}$ and let $\widehat{\mathcal{F}} = \{\widehat{\mathcal{S}}_\alpha\}_{\alpha\in\Gamma}$. Our next aim is to show that each family $\widehat{\mathcal{S}}_\alpha$ is a spherical isometry and that

$$\sum_{j\in J_\alpha} \hat{T}^*_{j,\alpha}\pi(X)\widehat{T}_{j,\alpha} = \pi(X)$$

for all $X \in C^*(\mathcal{T}(\mathcal{F}))$. For this purpose, fix $\alpha \in \Gamma$ and observe that $\sum_{j\in F} \hat{T}^*_{j,\alpha}\widehat{T}_{j,\alpha} \leq 1$ for each finite subset $F \subset J_\alpha$. Therefore

$$\sum_{j\in J_\alpha} \hat{T}^*_{j,\alpha}\widehat{T}_{j,\alpha} \leq 1.$$

Now, let $X \in C^*(\mathcal{T}(\mathcal{F}))$ and let $F \subset J_\alpha$ be a finite set. Then we see that

$$\rho(\sum_{j\in F} \hat{T}^*_{j,\alpha}\pi(X)\widehat{T}_{j,\alpha}) = \sum_{j\in F} T^*_{j,\alpha}\Phi(X)T_{j,\alpha}.$$

Taking weak*-limits in both sides, using the fact that ρ is a weak*-homeomorphism and using that ρ is isometric we infer that

$$\sum_{j\in J_\alpha} \hat{T}^*_{j,\alpha}\pi(X)\widehat{T}_{j,\alpha} = \pi(X)$$

for all $X \in C^*(\mathcal{T}(\mathcal{F}))$. In particular it follows that $\widehat{\mathcal{S}}_\alpha$ is indeed a spherical isometry. Moreover, using Lemma 2.8 we infer that all $\widehat{T}_{j,\alpha}$ belong to the center of $\pi(C^*(\mathcal{T}(\mathcal{F})))$, in particular they are commuting normal operators. Since $T_{j,\alpha} = V^*\widehat{T}_{j,\alpha}V$ and both $\mathcal{S}_\alpha$ and $\widehat{\mathcal{S}}_\alpha$ are spherical isometries it is easy to see that $\widehat{T}_{j,\alpha}V\mathcal{H} \subset V\mathcal{H}$ for all $\alpha \in \Gamma$ and $j \in J_\alpha$. This shows that the family $\mathcal{F}$ is subnormal, which proves item (2).

We will show now that $\widehat{\mathcal{F}}$ is the minimal normal extension of $\mathcal{F}$. For this purpose let $\mathcal{K}$ be the smallest reducing subspace for $\pi(C^*(\mathcal{F}))$ containing $V\mathcal{H}$. Let $\pi_{\mathcal{K}}: C^*(\mathcal{F}) \to \mathcal{B}(\mathcal{K})$ the *-representation defined by $\pi_{\mathcal{K}}(X) = P_{\mathcal{K}}\pi(X)|_{\mathcal{K}}$ where $P_{\mathcal{K}}$ denotes the orthogonal projection of $\widehat{\mathcal{H}}$ onto $\mathcal{K}$. We will show that the map defined by $\rho_{\mathcal{K}}(\pi(X)) = P_{\mathcal{K}}\pi(X)|_{\mathcal{K}}$ is a *-isomorphism of $\pi(C^*(\mathcal{T}(\mathcal{F})))$ onto the commutant $\pi_{\mathcal{K}}(C^*(\mathcal{F}))'$ of $\pi_{\mathcal{K}}(C^*(\mathcal{F}))$.

It is clear that $\rho_{\mathcal{K}}$ is a completely positive and completely contractive mapping. It takes values in $\pi_{\mathcal{K}}(C^*(\mathcal{F}))'$ because each $\widehat{T}_{j,\alpha}$ is in the center of $\pi(C^*(\mathcal{T}(\mathcal{F})))$ and because the space $\mathcal{K}$ is reducing for all $\widehat{T}_{j,\alpha}$. Let $\rho_{\mathcal{H}}: \pi_{\mathcal{K}}(C^*(\mathcal{F}))' \to \mathcal{B}(\mathcal{H})$ be defined by $\rho_{\mathcal{H}}(Y) = V^*YV$ for $Y \in \pi_{\mathcal{K}}(C^*(\mathcal{F}))'$. Then it is obvious that its image is included in $\mathcal{T}(\mathcal{F})$ and moreover $\rho = \rho_{\mathcal{H}}\rho_{\mathcal{K}}$ where $\rho: \pi(C^*(\mathcal{T}(\mathcal{F}))) \to \mathcal{B}(\mathcal{H})$ was defined above as $\rho(Y) = V^*YV$ for $Y \in \pi(C^*(\mathcal{T}(\mathcal{F})))$. Recall now that we already proved that ρ is completely isometric which implies that the mapping $\rho_{\mathcal{K}}$ is completely isometric. Therefore in order to show that $\rho_{\mathcal{K}}$ is onto, it suffices to show that the mapping $\rho_{\mathcal{H}}$ is one-to-one. Suppose therefore that $Y \in \pi_{\mathcal{K}}(C^*(\mathcal{F}))'$ is such that $\rho_{\mathcal{H}}(Y) = V^*YV = 0$. In order to show that $Y = 0$ it suffices, because $\pi_{\mathcal{K}}$ is a minimal dilation of Φ restricted to $C^*(\mathcal{F})$ and $\pi_{\mathcal{K}}(C^*(\mathcal{F}))$ is abelian and Y is in its commutant, to show that for any two finite families $\{T_{j_1,\alpha_1}, \ldots, T_{j_m,\alpha_m}\}$ and $\{T_{i_1,\beta_1}, \ldots, T_{i_n,\beta_n}\}$ we have $V^*\widehat{T}^*_{j_1,\alpha_1}\ldots\widehat{T}^*_{j_m,\alpha_m}Y\widehat{T}_{i_1,\beta_1}\ldots\widehat{T}_{i_n,\beta_n}V = 0$ and the latter equality follows immediately from the fact that $V\mathcal{H}$ is invariant for all $\widehat{\mathcal{F}}$. This shows that $\rho_{\mathcal{H}}$ is indeed one-to-one hence $\rho_{\mathcal{K}}$ is onto. Since, by a well known result of Kadison [Kad51], any completely isometric surjective unital mapping between two C^*- algebras is multiplicative it follows that $\rho_{\mathcal{K}}$ is indeed a *-isomorphism of $\pi(C^*(\mathcal{T}(\mathcal{F})))$ onto $\pi_{\mathcal{K}}(C^*(\mathcal{F}))'$ in particular the space $\mathcal{K}$ is invariant under $\pi(C^*(\mathcal{T}(\mathcal{F})))$. Since π is minimal, this shows that in fact we have that $\mathcal{K} = \widehat{\mathcal{H}}$. In particular this shows that $\pi_{\mathcal{K}}(X) = \pi(X)$ for all $X \in C^*(\mathcal{F})$. Moreover, the fact that $Ker\pi$ is the ideal generated by all operators of the form $XY - \Phi(XY)$ with $X, Y \in \mathcal{T}(\mathcal{F})$ follows by an easy induction argument on the length of an arbitrary product of elements from $\mathcal{T}(\mathcal{F})$ using the fact that $Ker\pi = Ker\Phi_0$ which equals $\{X - \Phi(X): X \in C^*(\mathcal{T}(\mathcal{F}))\}$ together with Theorem 2.1. This completes the proof of (3a), (3b), (3c) and (3d).

In order to prove (3e) we show first that $\Phi(C^*(\mathcal{F})) = C^*(\mathcal{F}) \cap \mathcal{T}(\mathcal{F})$. Since $\Phi^2 = \Phi$ it is enough to show that $\Phi(C^*(\mathcal{F})) \subset C^*(\mathcal{F})$. This inclusion follows easily from the fact that since $\pi_{\mathcal{K}}(C^*(\mathcal{F}))$ is abelian, Φ takes any finite product of $T_{j,\alpha}$'s and $T^*_{j,\alpha}$'s into a permutation of the same product having all the $T^*_{j,\alpha}$'s at the left and all the $T_{j,\alpha}$'s at the right.

Now we can easily prove that the kernel of $\pi_{\mathcal{K}}$ coincides with the ideal of $C^*(\mathcal{F})$ generated by all commutators $XY-YX$ with $X,Y\in C^*(\mathcal{F})\cap\mathcal{T}(\mathcal{F})$. First, since $\pi_{\mathcal{K}}(C^*(\mathcal{F}))$ is commutative, we have that any such commutator is in $\operatorname{Ker}(\pi_{\mathcal{K}})$. Let us denote by Φ_{00} the restriction of Φ to $C^*(\mathcal{F})$. We see from the proof of Theorem 2.2 that $\operatorname{Ker}\Phi_0=\operatorname{Ker}\pi$ therefore $\operatorname{Ker}\pi_{\mathcal{K}}=\operatorname{Ker}\Phi_{00}$ as well. On the other hand, since $\Phi_{00}^2=\Phi_{00}$ we see that $\operatorname{Ker}\Phi_{00}=\operatorname{Ran}(I-\Phi_{00})$. Now, if $X\in C^*(\mathcal{F})$ is a finite product of $T_{j,\alpha}$'s and $T_{j,\alpha}^*$'s it becomes obvious from the above description of $\Phi_{00}(X)$ that $X-\Phi_{00}(X)$ belongs to the ideal generated by all commutators $XY-YX$ with $X,Y\in C^*(\mathcal{F})\cap\mathcal{T}(\mathcal{F})$. This completes the proof of (3e).

An alternate proof of the fact that $Ker\pi_{\mathcal{K}}$ coincides with the commutator ideal in $C^*(\mathcal{F})$ can be based on Bunce' characterization of multiplicative functionals on C^*-algebras generated by commuting hyponormal operators in terms of their joint approximate point spectrum (see [Bun71]). Indeed, in our case, it can be easily shown that for any $\{T_1,\ldots,T_m\}\subset\mathcal{F}$, we have that $\sigma_{ap}(T_1,\ldots,T_m)$ equals $\sigma_{ap}(\pi(T_1,\ldots,\pi(T_m))$ where σ_{ap} stands for the joint approximate spectrum.

We now prove (3f). If $X\in\mathcal{B}(\mathcal{H})$ is such that X commutes with all operators from $\mathcal{F}$ then obviously X and X^*X belong to $\mathcal{T}(\mathcal{F})$. Suppose now that $X\in\mathcal{B}(\mathcal{H})$ is such that both X and X^*X belong to $\mathcal{T}(\mathcal{F})$. If $\hat{X}=\pi(X)$ then $\hat{X}$ commutes with all the normal extensions from $\widehat{\mathcal{F}}$ and $V^*\hat{X}V=X$. Moreover $\|\hat{X}\|=\|X\|$ so all we need to show is that $\hat{X}V\mathcal{H}\subset V\mathcal{H}$. For this purpose, we observe that since $X^*X\in\mathcal{T}(\mathcal{F})$ then

$$X^*X=V^*\pi(X^*X)V=V^*\hat{X}^*\hat{X}V.$$

Therefore if $\xi\in\mathcal{H}$ then

$$\|V^*\hat{X}V\xi\|=\|X\xi\|=\|\hat{X}V\xi\|$$

which implies that indeed $\hat{X}V\mathcal{H}\subset V\mathcal{H}$. This finishes the proof of (3f) and the proof of the theorem as well. $\square$

Let us note that both in the case of the unilateral shift S on $H^2(\mathbb{T})$, and in the more general case of the Szego n-tuple all the assertions from (2) and (3) are well known. We should refer here to the pioneering work of L.A. Coburn on the C^*-algebra generated by a single isometry; see [Co67] and [Co69]. In particular, in the first case (3d) and (3e) are classical results due to L. Coburn and R.G. Douglas, and moreover, the kernel of π at (3d) is the corresponding commutator ideal; see Chapter VII in [Do98] for a self-contained presentation of the classical theory and precise references to the original papers. In the case of the Szegő n-tuple on the unit sphere in $\mathbb{C}^n$ (3e) is proved in [Co73], and (3d) appears in [DJ77]. We mention also that in both these cases the commutator ideal appearing in (3e) was shown to coincide with the ideal of all compact operators on the corresponding Hardy space.

For the case of a commuting family of isometries, the existence of a commuting unitary extension was proved in [Ito58]. In this case, the commutant lifting at (3f) is quite straightforward once we assume Ito's result. These results hold true even on Banach spaces, see [Do69]. Exact sequences similar to that in (3e) for finite families of commuting isometries have been studied in [BCL78].

C^*-algebras generated by isometric representations of commuting semigroups have been studied mainly for semigroups of positive elements in ordered abelian groups, see for example [BC70], [Do72],[Mu87] or more recently [ALNR94] for a

study based on crossed products by endomorphisms, a line which has been intensively pursued in the last decade.

For the case of a single finite spherical isometry the existence of a normal extension along with a commutant lifting theorem were proved in [At90]; see also [AL96] for alternate proofs.

3. Toeplitz operators associated with function algebras

The following general framework is frequently used when dealing with Toeplitz operators on Hardy spaces. Let K be a compact Hausdorff space and let $C(K)$ denote the Banach algebra of all complex-valued continuous functions on K. Let $A \subset C(K)$ be a norm-closed subalgebra containing the constants and separating the points of K. Such algebras are called function algebras or uniform algebras (see [Ga69] for basics of uniform algebras). Let us consider a Borel probability measure m on K and let $supp(m)$ be its closed support. The generalized Hardy space $H^2(m)$ associated to A is the $L^2(m)$ closure of A. For any function $\phi \in L^\infty(m)$ the Toeplitz operator $T_\phi\colon B(H^2(m)) \to B(H^2(m))$ is defined by $T_\phi h = P_{H^2(m)}(\phi h)$ for $h \in H^2(m)$ where $P_{H^2(m)}$ is the orthogonal projection of $L^2(m)$ onto $H^2(m)$. We shall also consider the usual multiplication operators M_ϕ defined on $L^2(m)$ by $M_\phi f = \phi f$ for all $f \in L^2(m)$. Let $H^\infty(m)$ denote the intersection $H^2(m) \cap L^\infty(m)$ which is a weak*-closed subalgebra of $L^\infty(m)$. If $B \subset L^\infty(m)$ is any unital subalgebra, we shall denote by $\mathcal{T}(B)$ the C^*-subalgebra of $B(H^2(m))$ generated by all Toeplitz operators T_ϕ with $\phi \in B$ and by $\mathcal{C}(B)$ the closed ideal in $\mathcal{T}(B)$ generated by all operators of the form $T_\phi T_\psi - T_{\phi\psi}$ for arbitrary $\phi, \psi \in B$.

For our purposes we need to introduce the following definition. We shall say that a finite family of functions $F = \{\phi_1, \ldots, \phi_n\} \subset C(K)$ is a spherical multifunction if

$$\sum_{j=1}^{n} |\phi_j(x)|^2 = 1$$

for every $x \in K$.

We are now able to state the main result of this section.

Theorem 3.1. *Let K be a compact Hausdorff space and let $A \subset C(K)$ be a unital norm-closed subalgebra. Suppose there exists a family $\{F_\alpha\}_{\alpha \in \Gamma}$ of spherical multifunctions in $C(K)$ where each F_α is of the form $F_\alpha = \{\phi_{j,\alpha}\}_{j \in J_\alpha}$ with each $\phi_{j,\alpha} \in A$ and such that for each pair of distinct points $x, y \in K$ there exist an index $\alpha \in \Gamma$ and an index $j \in J_\alpha$ such that $\phi_{j,\alpha}(x) \neq \phi_{j,\alpha}(y)$. Then for any Borel probability measure m on K the following assertions hold true.*

(1) *A bounded operator $X \in B(H^2(m))$ is a Toeplitz operator if and only if it satisfies the following equations:*

$$\sum_{j \in J_\alpha} T^*_{\phi_{j,\alpha}} X T_{\phi_{j,\alpha}} = X$$

for all $\alpha \in \Gamma$ (in the case when A is the disc algebra and m is the Lebesgue measure on the unit circle we then retrieve the classical result of Brown-Halmos by taking as spherical multifunction $\phi(z) = z$).

(2) *A bounded operator $X \in B(H^2(m))$ is of the form $X = T_\psi$ for some $\psi \in H^\infty(m)$ if and only if it commutes with $T_{\phi_{j,\alpha}}$ for all $\alpha \in \Gamma$ and all $j \in J_\alpha$, if and*

only if it commutes with all T_ϕ with $\phi \in A$. The map $\psi \mapsto T_\psi$ is a Banach algebra isometric isomorphism between $H^\infty(m)$ and the commutant of the family $\{T_\phi : \phi \in A\}$. Moreover this commutant is a maximal abelian subalgebra of $B(H^2(m))$ and hence, for every $\phi \in H^\infty(m)$ we have that $\sigma(T_\phi) = \sigma_{H^\infty(m)}(\phi)$.

(3) *There exists a short exact sequence of C^*-algebras*

$$0 \to \mathcal{C}(L^\infty(m)) \hookrightarrow \mathcal{T}(L^\infty(m)) \xrightarrow{\pi} L^\infty(m) \to 0$$

such that $\pi(T_\phi) = \phi$ for all $\phi \in L^\infty(m)$. In particular the spectral inclusion $\operatorname{essran}(\phi) \subset \sigma(T_\phi)$ *holds true (in the case of the unit circle this is the classical theorem of Hartman and Wintner [HW54]) and we also have that*

$$\sigma(T_\phi) \subset conv(essran(\phi))$$

where conv denotes the convex hull.

(4) *There exists a short exact sequence*

$$0 \to \mathcal{C}(C(K)) \hookrightarrow \mathcal{T}(C(K)) \xrightarrow{\pi} C(supp(m)) \to 0$$

such that $\pi(T_\phi) = \phi$ on $supp(m)$. Moreover, in this case $\mathcal{C}(C(K))$ coincides always with the closed ideal in $\mathcal{T}(C(K))$ generated by all commutators $T_\phi T_\psi - T_\psi T_\phi$ with $\phi, \psi \in C(K)$.

Proof. Let us denote, for each $\alpha \in \Gamma$ and each $j \in J_\alpha$ by $T_{j,\alpha}$ the Toeplitz operator with symbol $\phi_{j,\alpha}$. Since each tuple $\{\phi_{j,\alpha}\}_{j \in J_\alpha}$ is a spherical multifunction it follows easily that in this case $\mathcal{S}_\alpha = \{T_{j,\alpha}\}_{\alpha \in \Gamma}$ is a spherical isometry and that $\mathcal{F} = \{\mathcal{S}_\alpha\}_{\alpha \in \Gamma}$ is a commuting family of spherical isometries in $B(H^2(m))$. The separation property imposed on these spherical multifunctions implies via the Stone-Weierstrass theorem that the C^*-algebra generated in $C(K)$ by the union of all families F_α with $\alpha \in \Gamma$ equals $C(K)$ itself. In turn this implies that the set $\widehat{\mathcal{F}}$ of all the corresponding multiplication operators $M_{\phi_{j,\alpha}}$ on $L^2(m)$ is the minimal normal extension of $\mathcal{F}$. Therefore, using Theorem 2.9 we infer that every operator $X \in B(H^2(m))$ satisfying equations (1) is the compression of a bounded operator Y in the commutant of all operators $M_{\phi_{j,\alpha}}$ therefore Y commutes with all multiplication operators M_ϕ with $\phi \in C(K)$ which implies that Y itself is a multiplication operator with some function $\psi \in L^\infty(m)$ which shows that X is a Toeplitz operator i.e. $X = T_\psi$. Conversely, any Toeplitz operator obviously satisfies these equations because $H^2(m)$ is invariant for all operators $M_{\phi_{j,\alpha}}$. This completes the proof of (1). Now, the proofs of (2), (3) and (4) follow easily from the previous remarks combined with Theorem 2.9 (the last assertion at (3) follows from the well-known fact that for any bounded operator T we have that $conv(\sigma(T)) \subset W(T)$ with equality for normal operators, where $W(T)$ stands for the numerical range of T; the case of the unit circle is due to A. Brown and P.R. Halmos [BH63]). $\square$

As a remark, it can be shown, on the same lines of the proof of Theorem 2.9, or using the results from [Bun71], that if the C^*-algebra generated by $H^\infty(m)$ in $L^\infty(m)$ coincides with $L^\infty(m)$ (equivalently if $H^\infty(m)$ separates the points in the maximal ideal space of $L^\infty(m)$) then $\mathcal{C}(L^\infty(m))$ coincides with the closed ideal of $\mathcal{T}(L^\infty(m))$ generated by all commutators $T_\phi T_\psi - T_\psi T_\phi$ with $\phi, \psi \in L^\infty(m)$ (for instance this is the case when K is the unit circle, A is the disc algebra and m is the Lebesgue measure on K; see [Do98]).

We also remark that a description of the character space of the quotient

$$\mathcal{T}(L^\infty(m))/\mathcal{C}(L^\infty(m))$$

valid for Hardy spaces over any function algebra was given in [Sun87].

Here follow two general examples of function algebras satisfying the hypotheses of Theorem 3.1. We emphasize that this result holds for *every* Borel probability measure on the corresponding Shilov boundaries. Toeplitz operators on such domains have been intensively studied in the literature; see [Up84] and [Up96].

Example 3.2. Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain and let $A(\Omega)$ be the algebra of all continuous functions on its closure and holomorphic on Ω. Let $K = \partial\Omega$ be the topological boundary of Ω which coincides in this case with the Shilov boundary of $A(\Omega)$ and let A be the set of all restrictions to $\partial\Omega$ of all functions from $A(\Omega)$. It follows from an embedding theorem for such domains (see Theorem 3 in [Lo85]) that there exist a natural number $N > 1$ and functions $f_1, \ldots, f_N$ in $A(\Omega)$ such that the function $F\colon \partial\Omega \to \mathbb{C}^N$ defined by $F(x) = (f_1(x), \ldots, f_N(x))$ is one-to-one and takes $\partial\Omega$ into the unit sphere in $\mathbb{C}^N$. This shows that it is a separating spherical multifunction for A and hence Theorem 3.1 applies in this case. In particular this applies to any bounded domain with C^2 boundary in the complex plane. For the case of finitely connected domains in $\mathbb{C}$ with analytic boundary, exact sequences of the form (3) and (4) were constructed in [Ab74].

Example 4.3. Let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain containing the origin and such that $e^{i\theta}\zeta \in \Omega$ whenever $\zeta \in \Omega$ and $\theta \in \mathbb{R}$. Let $A(\Omega)$ be as in the previous example. Let $K \subset \partial\Omega$ be the Shilov boundary of $A(\Omega)$ and let $\gamma = max\{|\zeta| \colon \zeta \in \overline{\Omega}\}$. It is then known that $K = \{z \in \partial\Omega\colon |z| = \gamma\}$ (see Theorem 6.5 in [Loo77]). Therefore the function $F(z) = z/\gamma$ is an imbedding of K into the unit sphere in $\mathbb{C}^n$ hence Theorem 3.1 applies in this case as well, taking A to be the algebra of all restrictions to K of functions from $A(\Omega)$. In particular we obtain that, given any Borel probability measure m on K, an operator $X \in B(H^2(m))$ is a Toeplitz operator if and only if

$$\sum_{j=1}^{n} T_{z_j}^* X T_{z_j} = \gamma^2 X.$$

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